Modern approaches to quantum gravity

Homework 13 Fall 2025

1. Hawking-Page in Three Dimensions

Recall the metric of the Euclidean BTZ black hole in AdS_3,

$$ds^2 = \ell^2 \left[\left(r^2 - 8GM\right) dt_E^2 + \frac{dr^2}{r^2 - 8GM} + r^2 d\phi^2 \right] . \tag{1}$$

The on-shell Euclidean action, including all boundary terms and counterterms is given by

$$S_E^{(bh)}(\beta) = -\frac{\pi^2 \ell c}{3\beta}, \tag{2}$$

where

$$c = \frac{3\ell}{2G}. \tag{3}$$

(a) Like in AdS_5, there is also a thermal AdS solution with the same boundary condition.[1] We will use a trick to compute its action. The trick is to note that (1) is a solid torus with boundary $S^1_{2\pi\ell} \times S^1_\beta$. (The subscript is the circumference of the S^1). The thermal circle S^1_β is 'filled in'.

Thermal AdS_3 is also a solid torus where instead the other circle $S^1_{2\pi\ell}$ is 'filled in'. Argue that this implies

$$S_E^{(th)}(\beta) = S_E^{(bh)}\left(\frac{4\pi^2\ell^2}{\beta}\right) = -\frac{c\beta}{12\ell}. \tag{4}$$

Comment: This is a special case of a modular transformation. It is a 'large'[2] conformal transformation acting on a torus, which roughly speaking relates a fat torus to a skinny torus.

(b) Sketch a plot of the free energies $F^{(bh)}$ and $F^{(th)}$. Find the critical temperature β_{crit} of the Hawking-Page phase transition, and write $\log Z(\beta)$ as a piecewise function.

(c) Find the thermodynamic entropy $S(\beta)$ for all $\beta > 0$.

(d) Find the energy $E(\beta)$ for all $\beta > 0$.

(e) Use part (d) in part (c) to find the entropy in the microcanonical ensemble, $S(E)$. (Be careful about what ranges of E your formulas apply to; in particular, you cannot find $S(E)$ for all E by this method.)

(f) Interpret your results in terms of the density of states in a 2d CFT dual to 3d gravity.

[1] Unlike AdS_5, there is only one black hole with temperature β.

[2] i.e., not continuously connected to the identity

2. Large and small black holes in general d

When one considers the thermal partition function of a gravitational system, one encounters the $(d+1)$-dimensional Euclidean AdS–Schwarzschild black hole with has a metric

$$ds^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{d-1}^2,$$

where $f(r) = 1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{L^2}$ and L is the AdS radius. This is a one-parameter family of solutions characterised by μ, which is related to the black hole mass. The case with $\mu = 0$ corresponds to the thermal AdS background, and it is always a solution of Einstein's equations.

(a) Using the periodicity trick introduced in a previous homework, compute the temperature of the corresponding thermal state as a function of the radius of the black hole horizon r_h. You should find

$$T = \frac{dr_h^2 + (d-2)L^2}{4\pi L^2 r_h}.$$

Comment on the difference between the role of the thermal AdS and black hole solutions.

(b) Show that there exists a minimal temperature T_{min} given by

$$T_{\text{min}} = \frac{1}{2\pi L}\sqrt{d(d-2)}.$$

Argue why for $T < T_{\text{min}}$, there is no solution with $\mu \neq 0$ for the spacetime metric introduced above, so thermal AdS is the only possible saddle.

(c) For $T \geq T_{\text{min}}$, there exist two black hole solutions (i.e. with $\mu \neq 0$). Find their radii as a function of temperature.

(d) Using the saddle point value of the free energy,

$$F = TS_{\text{on-shell}},$$

show that the difference between the entropy of a black hole and thermal AdS is give by

$$\Delta F = F_{BH} - F_{AdS} = \frac{r_h^{d-2}}{2\kappa^2}\text{Vol}(S^{d-1})\left(1 - \frac{r_h^2}{L^2}\right).$$

(e) Comment on the Hawking–Page phase transition happening in this system, and find the transition temperature T_{HP}. Show that small black hole solutions are never the dominant contribution to the partition function.

(f) Compute the specific heat of the three geometries. What happens if the geometry of a small black hole is prepared at temperature $T > T_{HP}$ and $T < T_{HP}$.
Hint: Use the fact that the renormalised free energy of thermal AdS is constant.

(g) What happens to the phase diagram as we take the limit $L \to \infty$?